4
Bosonic systems in phase space

In classical physics the state of a physical object and its dynamics can generally be illustrated by a time dependent probability density in phase space. The phase space is thereby spanned by the canonically conjugate variables as, for example, the position q and momentum p of a particle. The phase-space density $P(q, p, t)$ relates then to the probability $\mathrm{d}w$ at time t of observing the particle in the intervals $\mathrm{d}q$ and $\mathrm{d}p$, centered around the values q and p, via the typical expression $\mathrm{d}w = P(q, p, t)\mathrm{d}q\mathrm{d}p$.

Whereas this is perfectly appropriate in classical physics one encounters problems of interpretation in the quantum domain. Here Heisenberg's uncertainty relation $\Delta q \Delta p \geq \hbar/2$ prohibits one to consider the knowledge (i. e., observation) of both canonical variables at the same time with arbitrary precision. Proper probability densities in phase space, which are based on orthogonal projectors, may therefore be considered nonexistent. However, a complete description of the quantum mechanical state can still be obtained in phase space if one introduces phase-space functions in a wider sense. To do this, we may first generalize our description of a quantum mechanical system in order to include (classical) statistical uncertainties. This is readily obtained in terms of the statistical density operator, which includes both quantum and classical uncertainties in the inference of the properties of the considered physical object.

4.1
The statistical density operator

After the measurement of an observable with the observed outcome being O we encode our inferred knowledge of the physical state of the measured object in the form of the quantum state, which is chosen to be the eigenstate of the associated Hermitian operator $\hat{O}$ with eigenvalue O. Clearly this represents an idealized picture in that we assume a perfect detection of the observable. However, in general we may release these constraints to also allow statistical uncertainties in the measurement process itself. These uncertainties, being of classical nature, restrict the knowledge or information that can be gained

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in a measurement, since now a range of values of O may correspond to the measurement outcome.

To deal with such situations we have to incorporate classical statistics into our quantum mechanical description of the inferred state of a quantum mechanical object. Most naturally this is performed by turning to the statistical density operator which is a weighted sum of state projectors, with the weights having the properties of a probability distribution,

$$\hat{\varrho} = \sum_{\psi} P_{\psi} |\psi\rangle\langle\psi|. \tag{4.1}$$

Here P_{ψ} may be viewed as the probability of finding the system in the quantum state $|\psi\rangle$ and the sum goes over all possible or considered states. Clearly, the P_{ψ} being defined as probabilities have to satisfy the conditions

$$P_{\psi} \geq 0, \quad \sum_{\psi} P_{\psi} = 1. \tag{4.2}$$

In particular, a pure state $|\phi\rangle$ can be easily represented by a density operator by choosing the weights as $P_{\psi} = \delta_{\psi\phi}$. The density operator reduces then to the projector $\hat{\varrho} = |\phi\rangle\langle\phi|$.

The expectation value of a physical observable represented by the corresponding Hermitian operator $\hat{O}$ for the quantum state being described by a density operator reads as

$$\langle\hat{O}\rangle = \sum_{\psi} P_{\psi} \langle\psi|\hat{O}|\psi\rangle = \mathrm{Tr}(\hat{\varrho}\hat{O}). \tag{4.3}$$

From Eq. (4.3) we can see that when the system is prepared in a statistical mixture of quantum states (4.1), obtaining the expectation value of an observable is in general a two-fold procedure. Firstly the quantum-mechanical expectation values of the observable $\hat{O}$ must be calculated for the states $|\psi\rangle$ and secondly these expectation values must be averaged according to the probabilities P_{ψ} in the usual (classical) way.

Using the density operator itself as the observable, $\hat{O} = \hat{\varrho}$, we obtain from Eq. (4.3) the special result

$$\langle\hat{\varrho}\rangle = \sum_{\psi,\psi'} P_{\psi} P_{\psi'} |\langle\psi'|\psi\rangle|^2, \tag{4.4}$$

where by using the relation $|\langle\psi|\psi'\rangle| \leq 1$ we may derive from (4.4) the inequality

$$\langle\hat{\varrho}\rangle = \mathrm{Tr}\,\hat{\varrho}^2 \leq 1. \tag{4.5}$$

This inequality is in general regarded as a criterion for the statistical mixedness of a quantum state. When $\mathrm{Tr}\,\hat{\varrho}^2 = 1$ we have a pure state, whereas with decreasing value of this trace the state becomes more and more statistically mixed. To show this in more detail, let us consider an orthogonal and complete set of states $|j\rangle$, i. e.,

$$\langle j|j'\rangle = \delta_{jj'}, \quad \sum_j |j\rangle\langle j| = \hat{I}. \tag{4.6}$$

By the use of this representation, the density operator may be written in the form

$$\hat{\varrho} = \sum_{j,j'} \varrho_{jj'} |j\rangle\langle j'|, \tag{4.7}$$

where $\varrho_{jj'} = \langle j|\hat{\varrho}|j'\rangle$ are the matrix elements of the density operator in the chosen representation, or in short, the density matrix. We can easily prove that Eq. (4.3) may be rewritten as

$$\langle\hat{O}\rangle = \sum_{j,j'} \varrho_{jj'} \langle j'|\hat{O}|j\rangle. \tag{4.8}$$

When the density operator $\hat{\varrho}$ corresponds to a pure quantum state without any statistical mixedness, $\hat{\varrho} = |\phi\rangle\langle\phi|$, the corresponding density matrix in the representation of states $|j\rangle$ reads

$$\varrho_{jj'} = \langle j|\phi\rangle\langle\phi|j'\rangle. \tag{4.9}$$

From Eq. (4.8) ($\hat{O} \mapsto \hat{\varrho}$) in this case we obtain for the statistical mixedness

$$\mathrm{Tr}\,\hat{\varrho}^2 = \sum_{j,j'} |\langle j|\phi\rangle|^2 |\langle j'|\phi\rangle|^2 = 1, \tag{4.10}$$

due to the fact that the probability of finding the value j for the given state $|\phi\rangle$ is normalized to unity. In general one may observe that the moduli of the off-diagonal elements are smaller for a statistical mixture than those for a pure state, i. e., $|\rho_{jj'}| < \sqrt{|\rho_{jj}||\rho_{j'j'}|}$ for $j \neq j'$, which leads then to $\mathrm{Tr}\,\hat{\varrho}^2 < 1$.

As is well known, in the Schrödinger picture the state vector $|\psi(t)\rangle$ obeys the Schrödinger equation

$$i\hbar \frac{\mathrm{d}|\psi\rangle}{\mathrm{d}t} = \hat{H}|\psi\rangle, \tag{4.11}$$

where $\hat{H}$ is the Hamiltonian of the system under consideration. By considering the pure-state case $\hat{\varrho} = |\psi\rangle\langle\psi|$ it is obvious that the density operator then obeys the following equation of motion:

$$i\hbar \frac{\mathrm{d}\hat{\varrho}}{\mathrm{d}t} = [\hat{H}, \hat{\varrho}]. \tag{4.12}$$

The use of density operators is typically useful when dealing with a system composed of interacting subsystems, where only one of the subsystems is of interest. Let us consider, for example, two interacting systems, such as a radiation field coupled to an atomic system, with Hamiltonian $\hat{H}=\hat{H}_1+\hat{H}_2+\hat{H}_{\rm int}$. The Hilbert space of the total system is then the direct product of the Hilbert spaces of the two subsystems, that is, when $|j_1\rangle$ and $|j_2\rangle$ are forming complete sets of states of system 1 and 2, respectively, a complete set of states for the combined system is given by

$$|j_1, j_2\rangle = |j_1\rangle|j_2\rangle. \tag{4.13}$$

The expectation values of an observable $\hat{O}_k$ $(k=1,2)$ associated with only one of the subsystems, say subsystem 1, is obtained using Eq. (4.8) as

$$\langle \hat{O}_1\rangle = \mathrm{Tr}\,(\hat{\varrho}_1\hat{O}_1), \tag{4.14}$$

where the reduced density operator $\hat{\varrho}_1$, which describes subsystem 1 alone, is given by the trace with respect to subsystem 2,

$$\hat{\varrho}_1 = \mathrm{Tr}_2\ \hat{\varrho} = \sum_{j_2}\langle j_2|\hat{\varrho}|j_2\rangle. \tag{4.15}$$

Note that, even when the overall system is in a pure quantum state, the subsystems, as represented by their reduced density operators, in general are not. If the overall system is closed, i. e., isolated from its environment, the corresponding density operator of the system will obey an equation of motion of the type (4.12). However, the dynamics of a system which is interacting with its environment, cannot in general be described by such a unitary time evolution.

4.2 Phase-space functions

Let $\hat{O}$ be an operator that may be a function of $\hat{a}$ and $\hat{a}^\dagger$, $\hat{O}=\hat{f}(\hat{a},\hat{a}^\dagger)$, and let us consider its expectation value when the system is described by the density operator $\hat{\varrho}$,

$$\langle \hat{O}\rangle = \langle \hat{f}(\hat{a},\hat{a}^\dagger)\rangle = \mathrm{Tr}[\hat{\varrho}\,\hat{f}(\hat{a},\hat{a}^\dagger)]. \tag{4.16}$$

To perform the trace, a set of complete quantum states has to be chosen. To arrive at a phase-space description it is convenient to choose the coherent states $|\alpha\rangle$ as basis states. From Eq. (3.71) we obtain the coherent-state representation of the density operator as

$$\hat{\varrho} = \frac{1}{\pi^2}\int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta\, \varrho(\alpha,\beta)|\alpha\rangle\langle\beta|, \tag{4.17}$$

where the density matrix in the coherent-state basis is given by

$$\varrho(\alpha, \beta) = \langle \alpha | \hat{\varrho} | \beta \rangle. \tag{4.18}$$

Inserting Eq. (4.17) into Eq. (4.16), we obtain the expectation value $\langle \hat{f}(\hat{a}, \hat{a}^\dagger) \rangle$ in the form of

$$\langle \hat{f}(\hat{a}, \hat{a}^\dagger) \rangle = \frac{1}{\pi^2} \int d^2\alpha \int d^2\beta \, \varrho(\alpha, \beta) \langle \beta | \hat{f}(\hat{a}, \hat{a}^\dagger) | \alpha \rangle. \tag{4.19}$$

By substituting $\hat{a}, \hat{a}^\dagger \mapsto \alpha, \alpha^*$ we may arrive at classical statistics, where the operator function which corresponds to the operator $\hat{O}$ will turn out to be a function in the phase-space spanned by the complex number α,

$$\hat{O} = \hat{f}(\hat{a}, \hat{a}^\dagger) \;\mapsto\; O = f(\alpha, \alpha^*) \equiv f(\alpha). \tag{4.20}$$

In this case the expectation value of the quantity O is obtained by the usual statistical averaging as

$$\langle O \rangle_{\mathrm{cl}} = \int d^2\alpha \, P_{\mathrm{cl}}(\alpha) \, f(\alpha), \tag{4.21}$$

where the phase-space function $P_{\mathrm{cl}}(\alpha)$ is the classical probability density of observing the complex field amplitude α, which fully describes the (classical) state of the system. The question arises as to whether or not the quantum-mechanical expectation value (4.19) may be represented in a form similar to that of classical theory, Eq. (4.21). As we shall see below, Eq. (4.19) can indeed be rewritten in a form which formally looks like Eq. (4.21), provided that the operator under study, $\hat{O}$, is ordered in certain ways with respect to the operators $\hat{a}$, $\hat{a}^\dagger$. However, the phase-space functions found in this way cannot be viewed, in general, as being probability distribution functions.

4.2.1 Normal ordering: The P function

To arrive at one of these phase-space functions, let us assume that, by means of the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, the operator $\hat{O} = \hat{f}(\hat{a}, \hat{a}^\dagger)$ is put into normal order. Normal order means in this context that in the resulting expression all the creation operators are positioned left of the annihilation operators. That is, if $\hat{f}^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger)$ is the resulting expression in normal order, we have the equivalence

$$\hat{f}(\hat{a}, \hat{a}^\dagger) \equiv \hat{f}^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger). \tag{4.22}$$

Furthermore, we may now define the associated c-number function $f^{(\mathrm{N})}(\alpha) \equiv f^{(\mathrm{N})}(\alpha, \alpha^*)$ by substituting in $\hat{f}^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger)$ for the operators $\hat{a}$ and $\hat{a}^\dagger$ the complex

c numbers α and α^*, respectively. Obviously, $f^{(\mathrm{N})}(\alpha)$ is simply the diagonal matrix element of $\hat{f}(\hat{a}, \hat{a}^\dagger)$ with the coherent state $|\alpha\rangle$:

$$f^{(\mathrm{N})}(\alpha) = \langle\alpha|\hat{f}^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger)|\alpha\rangle = \langle\alpha|\hat{f}(\hat{a}, \hat{a}^\dagger)|\alpha\rangle. \tag{4.23}$$

Having this c-number function at hand, we now intend to express the operator $\hat{f}(\hat{a}, \hat{a}^\dagger)$ in terms of $f^{(\mathrm{N})}(\alpha)$ and other suitably chosen normally ordered operator functions. For this purpose, let us inspect the identity

$$f^{(\mathrm{N})}(\alpha) = \int \mathrm{d}^2\beta\, \delta(\alpha-\beta) f^{(\mathrm{N})}(\beta), \tag{4.24}$$

where $\delta(\alpha)$ is the usual two-dimensional Dirac δ function for real and imaginary parts of the argument, i. e., $(\alpha = \alpha' + i\alpha'')$,

$$\delta(\alpha) = \delta(\alpha')\delta(\alpha'') = \frac{1}{4\pi^2}\int \mathrm{d}x \int \mathrm{d}y\, \exp[i(\alpha' y + \alpha'' x)], \tag{4.25}$$

Substituting $\gamma = \pm(iy - x)/2$, we may rewrite the delta function (4.25) in a more convenient form, as an integral over the complex variable γ,

$$\delta(\alpha) = \frac{1}{\pi^2}\int \mathrm{d}^2\gamma\, \exp(\alpha^*\gamma - \alpha\gamma^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\gamma\, \exp(\alpha\gamma^* - \alpha^*\gamma). \tag{4.26}$$

Inserting this expression for the delta function in Eq. (4.24), we obtain

$$f^{(\mathrm{N})}(\alpha) = \frac{1}{\pi^2}\int \mathrm{d}^2\beta \int \mathrm{d}^2\gamma\, f^{(\mathrm{N})}(\beta) \exp[(\alpha-\beta)^*\gamma - (\alpha-\beta)\gamma]. \tag{4.27}$$

Going from the associated c-number function $f^{(\mathrm{N})}(\alpha)$ back to the operator function $\hat{f}^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger)$, i. e., re-substituting $\alpha, \alpha^* \mapsto \hat{a}, \hat{a}^\dagger$, we see from Eq. (4.27) that the operator $\hat{f}^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger)$ may be represented as

$$\hat{f}^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger) = \frac{1}{\pi^2}\int \mathrm{d}^2\beta \int \mathrm{d}^2\gamma\, f^{(\mathrm{N})}(\beta)\, \exp\left[(\hat{a}^\dagger - \beta^*)\gamma\right] \exp\left[-(\hat{a}-\beta)\gamma^*\right]. \tag{4.28}$$

This substitution is allowed since, before replacing the c numbers by operators, we have factored the exponential function in order to obtain a normally ordered representation where the $\hat{a}^\dagger$ are located left of the $\hat{a}$.

Next, we introduce an operator-valued version of the Dirac δ function in straightforward generalization of Eq. (4.26),

$$\hat{\delta}(\hat{a} - \alpha) = \frac{1}{\pi^2}\int \mathrm{d}^2\beta\, \exp[(\hat{a}^\dagger - \alpha^*)\beta - (\hat{a}-\alpha)\beta^*], \tag{4.29}$$

which may be also written as a Fourier transform of the displacement operator (3.44),

$$\hat{\delta}(\hat{a} - \alpha) = \frac{1}{\pi^2}\int \mathrm{d}^2\beta\, \hat{D}(\beta) \exp(\alpha\beta^* - \alpha^*\beta). \tag{4.30}$$

Applying the normal-ordering prescription $\mathcal{N}$ onto the displacement operator we obtain[1]

$$\mathcal{N}\hat{D}(\alpha) \equiv :\hat{D}(\alpha): = e^{\hat{a}^\dagger\alpha} e^{-\hat{a}\alpha^*}. \tag{4.31}$$

Using the relations (4.30) and (4.31), we may then rewrite Eq. (4.28) as

$$\hat{f}(\hat{a}, \hat{a}^\dagger) \equiv \hat{f}^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger) = \int \mathrm{d}^2\alpha\, f^{(\mathrm{N})}(\alpha)\mathcal{N}\hat{\delta}(\hat{a} - \alpha). \tag{4.32}$$

We now take the quantum-mechanical expectation value of both sides of Eq. (4.32) and obtain, on assuming that this operation and the integration can be interchanged, the sought result:

$$\langle \hat{f}(\hat{a}, \hat{a}^\dagger)\rangle = \langle \hat{f}^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger)\rangle = \int \mathrm{d}^2\alpha\, P^{(\mathrm{N})}(\alpha)\, f^{(\mathrm{N})}(\alpha), \tag{4.33}$$

where the phase-space function $P^{(\mathrm{N})}(\alpha)$ is the expectation value of the operator delta function in normal order,

$$P^{(\mathrm{N})}(\alpha) = \langle :\hat{\delta}(\hat{a} - \alpha): \rangle. \tag{4.34}$$

Although Eqs (4.21) and (4.33) bear a great resemblance, there are essential differences between the two equations. Firstly, the c-number function $f^{(\mathrm{N})}(\alpha)$ in Eq. (4.33) is associated with the operator $\hat{f}(\hat{a}, \hat{a}^\dagger)$ being transformed into its equivalent normally ordered form. That is, the complex numbers α and α^* are substituted for the operators in the operator function $\hat{f}^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger)$ and not in the original form of the operator function $\hat{f}(\hat{a}, \hat{a}^\dagger)$. Secondly, in general, the function $P^{(\mathrm{N})}(\alpha)$ cannot be regarded as a proper probability distribution function: $P^{(\mathrm{N})}(\alpha)$ can attain negative values that are not interpretable as probability densities and furthermore $P^{(\mathrm{N})}(\alpha)$ need not be a well-behaved function [for reviews, see Klauder and Sudarshan (1968); Peřina (1991)]. Notwithstanding these facts, in any case the function $P^{(\mathrm{N})}(\alpha)$ is normalized,

$$\int d^2\alpha\, P^{(\mathrm{N})}(\alpha) = 1, \tag{4.35}$$

which may readily be proved from Eq. (4.33) by choosing $\hat{f}(\hat{a}, \hat{a}^\dagger) = \hat{I}$. The quantum-state representation based on the phase-space function $P^{(\mathrm{N})}(\alpha)$ is called the Glauber–Sudarshan representation [Glauber (1963); Sudarshan (1963)], $P^{(\mathrm{N})}(\alpha)$ being also called the P function, where a frequently used abbreviated notation is

$$P(\alpha) \equiv P^{(\mathrm{N})}(\alpha). \tag{4.36}$$

1) The process of normal ordering as described by $\mathcal{N}$ is not to be confused with a normally ordered, equivalent representation of an operator, such as represented by the relation $\hat{f}^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger) = \hat{f}(\hat{a}, \hat{a}^\dagger)$. $\mathcal{N}$ is not an equivalence operation, i. e., $\mathcal{N}\hat{f}(\hat{a}, \hat{a}^\dagger) \neq \hat{f}(\hat{a}, \hat{a}^\dagger)$.

The Glauber–Sudarshan representation is of special importance in the context of photodetection where the appearing expectation values contain normally ordered moments and correlations. Moreover, although the P function may be an ill-behaving function, this distribution proves to be very useful for formal derivations in connection with operator expectation values.

4.2.2
Anti-normal and symmetric ordering: The Q and the W function

The applicability of the concept of phase-space functions as outlined above, is of course not restricted to the case of normal order. For example, if the operator $\hat{O}$ can be put in anti-normal order by use of the commutator relation $[\hat{a},\hat{a}^\dagger]=1$, $\hat{O}=\hat{f}^{(\mathrm{A})}(\hat{a},\hat{a}^\dagger)$, we may introduce the associated c-number function $f^{(\mathrm{A})}(\alpha)$ by substituting in $\hat{f}^{(\mathrm{A})}(\hat{a},\hat{a}^\dagger)$ for the operators $\hat{a}$ and $\hat{a}^\dagger$ the c numbers α and α^*, respectively. Performing manipulations analogous to those leading to Eq. (4.28) now yields the corresponding expression in anti-normal order,

$$\hat{f}(\hat{a},\hat{a}^\dagger) = \frac{1}{\pi^2}\int \mathrm{d}^2\beta \int \mathrm{d}^2\gamma\, f^{(\mathrm{A})}(\beta)\exp[-(\hat{a}-\beta)\gamma^*]\exp[(\hat{a}^\dagger-\beta^*)\gamma]. \quad (4.37)$$

Hence, instead of the normally ordered delta operator we now use the anti-normally ordered version,

$$\mathcal{A}\hat{\delta}(\hat{a}-\alpha) \equiv \ \ddagger\hat{\delta}(\hat{a}-\alpha)\ddagger = \frac{1}{\pi^2}\int \mathrm{d}^2\beta\ \ddagger\hat{D}(\beta)\ddagger\exp(\alpha\beta^*-\alpha^*\beta), \quad (4.38)$$

so that the anti-normally ordered displacement operator reads

$$\ddagger\hat{D}(\alpha)\ddagger = e^{-\hat{a}\alpha^*}e^{\hat{a}^\dagger\alpha}. \quad (4.39)$$

We then obtain in analogy with Eq. (4.32)

$$\hat{f}(\hat{a},\hat{a}^\dagger) = \int \mathrm{d}^2\alpha\, f^{(\mathrm{A})}(\alpha)\mathcal{A}\hat{\delta}(\hat{a}-\alpha), \quad (4.40)$$

from which the relation for the expectation value is derived as

$$\langle\hat{f}(\hat{a},\hat{a}^\dagger)\rangle = \langle\hat{f}^{(\mathrm{A})}(\hat{a},\hat{a}^\dagger)\rangle = \int \mathrm{d}^2\alpha\, P^{(\mathrm{A})}(\alpha)f^{(\mathrm{A})}(\alpha). \quad (4.41)$$

The phase-space function $P^{(\mathrm{A})}(\alpha)$, which is called the Husimi Q function, is the expectation value of the operator delta function in anti-normal order,

$$Q(\alpha) \equiv P^{(\mathrm{A})}(\alpha) = \langle\ddagger\hat{\delta}(\alpha-\hat{a})\ddagger\rangle. \quad (4.42)$$

On the other hand, taking the expectation value of the original operator delta function as defined by Eq. (4.29) obviously yields the phase space function suitable for averaging symmetrically ordered quantities,

$$\langle\hat{f}(\hat{a},\hat{a}^\dagger)\rangle = \langle\hat{f}^{(\mathrm{S})}(\hat{a},\hat{a}^\dagger)\rangle = \int \mathrm{d}^2\alpha\, P^{(\mathrm{S})}(\alpha)f^{(\mathrm{S})}(\alpha), \quad (4.43)$$

where

$$W(\alpha) \equiv P^{(\mathrm{S})}(\alpha) = \langle \hat{\delta}(\alpha - \hat{a}) \rangle \tag{4.44}$$

is called the Wigner function, and $f^{(\mathrm{S})}(\alpha)$ is the c-number function associated with the operator $\hat{f}(\hat{a}, \hat{a}^\dagger)$ in symmetrical order. We will not give more details here, but instead, in the following we consider the more general case of so-called s ordering.

4.2.3
Parameterized phase-space functions

The phase-space functions considered above may be regarded as certain special cases of an operator $\hat{O}$ being put in a chosen order [Cahill and Glauber (1969); for a review, see also Peřina (1991)]. To generalize the concept of operator ordering, we may start with the displacement operator and define its s-ordered representation by

$$\hat{D}(\alpha; s) = \hat{D}(\alpha) e^{|\alpha|^2 s/2}, \tag{4.45}$$

which implies that

$$\hat{D}(\alpha; s) = \exp\left[\tfrac{1}{2}(s - s')|\alpha|^2\right] \hat{D}(\alpha; s'). \tag{4.46}$$

The case $s=0$ is then considered as symmetrical ordering, since we obtain the original displacement operator,

$$\hat{D}(\alpha; 0) = \hat{D}(\alpha). \tag{4.47}$$

Moreover, comparing the expression (4.45) for the values $s=\pm 1$ with Eqs (4.31) and (4.39), we arrive at the following relations:

$$\hat{D}(\alpha; 1) = e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} = \,:\hat{D}(\alpha):\,, \tag{4.48}$$

$$\hat{D}(\alpha; -1) = e^{-\alpha^* \hat{a}} e^{\alpha \hat{a}^\dagger} = \,\ddagger \hat{D}(\alpha) \ddagger\,. \tag{4.49}$$

From Eqs (3.44)–(3.46) we see that choosing $s=0$, $s=1$ and $s=-1$ corresponds to putting the displacement operator in symmetrical, normal and anti-normal order, respectively. It should be pointed out that more general ordering procedures can be introduced, which unify s-ordering with other orderings such as standard and anti-standard ordering [Agarwal and Wolf (1968)].[2]

Let us now assume that the operator $\hat{O} = \hat{f}(\hat{a}, \hat{a}^\dagger)$ can be represented in any s-order $-1 \leq s \leq 1$ as[3]

$$\hat{f}(\hat{a}, \hat{a}^\dagger) = \int \mathrm{d}^2\alpha\, f(\alpha; s) \hat{\delta}(\hat{a} - \alpha; s), \tag{4.50}$$

2) For a detailed treatment of these unified ordering methods, see Agarwal and Wolf (1970).

3) Note that the value of s is not necessarily restricted to the interval $-1, \ldots, 1$.

where we have defined via Eq. (4.45) the general s-ordered operator delta function

$$\hat{\delta}(\hat{a}-\alpha;s) = \frac{1}{\pi^2}\int \mathrm{d}^2\beta\, \hat{D}(\beta;s)\exp(\alpha\beta^* - \alpha^*\beta) \tag{4.51}$$

[for details about the existence of the representation (4.50), see Cahill and Glauber (1969)]. Obviously, the c-number function $f(\alpha;s)$ associated with the operator $\hat{f}(\hat{a},\hat{a}^\dagger)$ in the chosen order, reduces in the special cases $s=0,\pm1$ to the familiar expressions $f(\alpha;0)\equiv f^{(\mathrm{S})}(\alpha)$, $f(\alpha;1)\equiv f^{(\mathrm{N})}(\alpha)$ and $f(\alpha;-1)\equiv f^{(\mathrm{A})}(\alpha)$. With the help of Eq. (4.50) the expectation value of an arbitrary operator $\hat{O}=\hat{f}(\hat{a},\hat{a}^\dagger)$ may now be written as

$$\langle\hat{O}\rangle = \langle\hat{f}(\hat{a},\hat{a}^\dagger)\rangle = \int \mathrm{d}^2\alpha\, P(\alpha;s) f(\alpha;s), \tag{4.52}$$

where the s-parameterized phase-space function $P(\alpha;s)$ is defined as[4]

$$P(\alpha;s) = \langle\hat{\delta}(\hat{a}-\alpha;s)\rangle. \tag{4.53}$$

It is often useful to represent the s-ordered operator delta function in a somewhat different form. Expressing $\hat{D}(\alpha;s)$ in terms of $\hat{D}(\alpha;s')$ in Eq. (4.51) according to Eq. (4.46) and applying, with respect to $\hat{D}(\alpha;s')$, the inverse of Eq. (4.51), we may write

$$\begin{aligned}\hat{\delta}(\hat{a}-\alpha;s) &= \frac{1}{\pi^2}\int \mathrm{d}^2\beta\, \exp\Big[\alpha\beta^* - \alpha^*\beta + \tfrac{1}{2}(s-s')|\beta|^2\Big]\,\hat{D}(\beta;s')\\ &= \frac{1}{\pi^2}\int \mathrm{d}^2\beta\, \exp\Big[\alpha\beta^* - \alpha^*\beta + \tfrac{1}{2}(s-s')|\beta|^2\Big]\\ &\qquad\times\int \mathrm{d}^2\gamma\, \exp(\beta\gamma^* - \beta^*\gamma)\hat{\delta}(\hat{a}-\gamma;s'). \end{aligned}\tag{4.54}$$

For $s\le s'$ the integration over β can be performed[5] to obtain

$$\hat{\delta}(\hat{a}-\alpha;s) = \frac{2}{\pi(s'-s)}\int \mathrm{d}^2\gamma\, \exp\left(-\frac{2|\alpha-\gamma|^2}{s'-s}\right)\hat{\delta}(\hat{a}-\gamma;s'). \tag{4.55}$$

For $s'=1$ the operator $\hat{\delta}(\hat{a}-\gamma;1)$ is the normally ordered form of the delta-function operator, and therefore the γ integration yields

$$\hat{\delta}(\hat{a}-\alpha;s) = \frac{2}{\pi(1-s)}\ :\exp\left[-\frac{2(\hat{a}^\dagger-\alpha^*)(\hat{a}-\alpha)}{1-s}\right]:, \tag{4.56}$$

which with the help of Eqs (3.47) and (3.48) may be rewritten in the form

$$\hat{\delta}(\hat{a}-\alpha;s) = \frac{2}{\pi(1-s)}\ :\exp\left[-\frac{2\hat{n}(\alpha)}{1-s}\right]:, \tag{4.57}$$

4) Recall that $W(\alpha)\equiv P^{(\mathrm{S})}(\alpha)\equiv P(\alpha;0)$, $P(\alpha)\equiv P^{(\mathrm{N})}(\alpha)\equiv P(\alpha;1)$ and $Q(\alpha)\equiv P^{(\mathrm{A})}(\alpha)\equiv P(\alpha;-1)$.

5) Note that the more general condition is Re $(s-s')\le 0$.

with the displaced number operator being defined as

$$\hat{n}(\alpha) = \hat{D}(\alpha)\hat{n}\hat{D}^{\dagger}(\alpha). \tag{4.58}$$

Equation (4.57) can be further evaluated to obtain

$$\begin{aligned}
\hat{\delta}(\hat{a}-\alpha;s) &= \frac{2}{\pi(1-s)}\ :\exp\left[\frac{s+1}{s-1}\,\hat{n}(\alpha)\right]\exp[-\hat{n}(\alpha)]: \\
&= \frac{2}{\pi(1-s)}\sum_{n=0}^{\infty}\left(\frac{s+1}{s-1}\right)^{n} : \frac{[\hat{n}(\alpha)]^{n}}{n!}\ \exp[-\hat{n}(\alpha)]: \\
&= \frac{2}{\pi(1-s)}\sum_{n=0}^{\infty}\left(\frac{s+1}{s-1}\right)^{n}\hat{D}(\alpha)\ : \frac{\hat{n}^{n}}{n!}\,e^{-\hat{n}}:\ \hat{D}^{\dagger}(\alpha).
\end{aligned} \tag{4.59}$$

Since $|\langle n|\alpha\rangle|^2$ as given by Eq. (3.60) is the c-number function associated with $|n\rangle\langle n|$ in normal order, we find that

$$:\frac{\hat{n}^{n}}{n!}\,e^{-\hat{n}}: = |n\rangle\langle n|, \tag{4.60}$$

and Eq. (4.59) can be rewritten as

$$\hat{\delta}(\hat{a}-\alpha;s) = \frac{2}{\pi(1-s)}\sum_{n=0}^{\infty}\left(\frac{s+1}{s-1}\right)^{n}\hat{D}(\alpha)|n\rangle\langle n|\hat{D}^{\dagger}(\alpha), \tag{4.61}$$

equivalently

$$\hat{\delta}(\hat{a}-\alpha;s) = \frac{2}{\pi(1-s)}\,\hat{D}(\alpha)\left(\frac{s+1}{s-1}\right)^{\hat{n}}\hat{D}^{\dagger}(\alpha). \tag{4.62}$$

For $s=0$ the s-ordered operator delta function $\hat{\delta}(\hat{a}-\alpha;0)$ reduces to the ordinary (i. e., symmetrically ordered) operator delta function $\hat{\delta}(\hat{a}-\alpha)$ defined by Eq. (4.29), and from Eq. (4.62) it then follows that

$$\hat{\delta}(\hat{a}-\alpha) = 2\pi^{-1}\hat{D}(\alpha)(-1)^{\hat{n}}\hat{D}^{\dagger}(\alpha) = 2\pi^{-1}(-1)^{\hat{n}(\alpha)}. \tag{4.63}$$

That is, the operator delta function is given (apart from the factor $2/\pi$) by the displaced parity operator and the Wigner function is simply the expectation value of that operator:

$$W(\alpha) \equiv P(\alpha;0) = 2\pi^{-1}\big\langle\hat{D}(\alpha)(-1)^{\hat{n}}\hat{D}^{\dagger}(\alpha)\big\rangle = 2\pi^{-1}\big\langle(-1)^{\hat{n}(\alpha)}\big\rangle. \tag{4.64}$$

Equation (4.64) reveals that the Wigner function cannot be regarded as a probability distribution in general, because it may attain negative values,

$$-2\pi^{-1} \le W(\alpha) \le 2\pi^{-1}. \tag{4.65}$$

For $s=-1$ Eq. (4.61) reduces to[6]

$$\hat{\delta}(\hat{a}-\alpha;-1) = \pi^{-1}|\alpha\rangle\langle\alpha|. \tag{4.66}$$

Hence, the Q function is given (apart from the factor π^{-1}) by the c-number function associated with the density operator in normal order,

$$Q(\alpha) \equiv P(\alpha;-1) = \pi^{-1}\langle\alpha|\hat{\varrho}|\alpha\rangle, \tag{4.67}$$

from which it follows that

$$0 \le Q(\alpha) \le \pi^{-1}. \tag{4.68}$$

It is worth noting that, although the Q function does not allow an interpretation as a probability distribution of the complex amplitude α in the sense of classical theory, it has all the properties of a probability distribution. As can be seen from Eq. (4.61), for $s=1$ the corresponding operator delta function $\hat{\delta}(\hat{a}-\alpha;1)$ is not bounded, and therefore the P function $P(\alpha)\equiv P(\alpha;1)$ is not necessarily a well-behaved phase-space function.

It should be noted that Eq. (4.54) can be used to express the function $P(\alpha;s)$, Eq. (4.53), in terms of another function $P(\alpha;s')$:

$$P(\alpha;s) = \frac{1}{\pi^2}\int \mathrm{d}^2\beta\, \exp\left[\alpha\,\beta^* - \alpha^*\beta + \tfrac{1}{2}(s-s')|\beta|^2\right]\int \mathrm{d}^2\gamma\, \exp(\beta\gamma^* - \beta^*\gamma)P(\gamma;s'). \tag{4.69}$$

For $s \le s'$ (or $\mathrm{Re}\,s \le \mathrm{Re}\,s'$) the integration over β can again be performed to obtain

$$P(\alpha;s) = \frac{2}{\pi(s'-s)}\int \mathrm{d}^2\gamma\, P(\gamma;s')\exp\left(-\frac{2|\alpha-\gamma|^2}{s'-s}\right). \tag{4.70}$$

In the opposite case when $s'<s$ (or $\mathrm{Re}\,s'<\mathrm{Re}\,s$), the integration over γ should be done first in Eq. (4.69) in order to avoid having to deal with singular expressions.

4.3 Operator expansion in phase space

Equation (4.50) can be viewed as an expansion of an operator $\hat{f}(\hat{a},\hat{a}^\dagger)$ in terms of the generalized projectors $\hat{\delta}(\hat{a}-\alpha;s)$. Whereas for $s=0,\pm1$ it is clear, in principle, how to obtain the associated c-number functions $f(\alpha;s)$, for arbitrary values of s we still require a recipe.

6) Note that for $s=-1$, from comparison of Eq. (4.57) and (4.66), it follows that $|\alpha\rangle\langle\alpha| = :\exp[-\hat{n}(\alpha)]:$.

4.3.1 Orthogonalization relations

For the purpose of deriving such a prescription we may consider the following relation obtained by taking the trace of Eq. (4.50) multiplied by $\hat{\delta}(\hat{a}-\beta;-s)$,

$$\mathrm{Tr}[\hat{f}(\hat{a},\hat{a}^\dagger)\hat{\delta}(\hat{a}-\beta;-s)] = \int \mathrm{d}^2\alpha\, f(\alpha;s)\,\mathrm{Tr}[\hat{\delta}(\hat{a}-\alpha;s)\hat{\delta}(\hat{a}-\beta;-s)]. \quad (4.71)$$

To obtain the trace on the right-hand side of Eq. (4.71) we first calculate the trace (via Fourier transformation) contained therein over the displacement operators. Using Eqs (3.53) and (4.45) we may write

$$\begin{aligned}\mathrm{Tr}[\hat{D}(\alpha;s)\hat{D}(\beta;-s)] &= \exp\left[\tfrac{1}{2}s(|\alpha|^2-|\beta|^2)\right]\mathrm{Tr}[\hat{D}(\alpha)\hat{D}(\beta)]\\ &= \exp\left[\tfrac{1}{2}s(|\alpha|^2-|\beta|^2)+i\,\mathrm{Im}(\alpha\beta^*)\right]\mathrm{Tr}[\hat{D}(\alpha+\beta)].\end{aligned} \quad (4.72)$$

We calculate the trace of the displacement operator using the coherent-state basis and applying Eqs (3.54) and (4.26):

$$\begin{aligned}\mathrm{Tr}[\hat{D}(\alpha)] &= \frac{1}{\pi}\int \mathrm{d}^2\beta\,\langle\beta|\hat{D}(\alpha)|\beta\rangle\\ &= e^{-|\alpha|^2/2}\,\frac{1}{\pi}\int \mathrm{d}^2\beta\,\exp(\alpha\beta^*-\alpha^*\beta) = \pi\delta(\alpha).\end{aligned} \quad (4.73)$$

Hence, Eq. (4.72) takes the form of

$$\mathrm{Tr}[\hat{D}(\alpha;s)\hat{D}(\beta;-s)] = \pi\delta(\alpha+\beta), \quad (4.74)$$

and combining Eqs (4.51) and (4.74) yields

$$\mathrm{Tr}[\hat{\delta}(\hat{a}-\alpha;s)\hat{\delta}(\hat{a}-\beta;-s)] = \pi^{-1}\delta(\alpha-\beta). \quad (4.75)$$

Note that Eq. (4.73) implies that

$$\mathrm{Tr}[\hat{\delta}(\hat{a}-\alpha;s)] = \pi^{-1}. \quad (4.76)$$

Inserting the orthogonalization relation (4.75) into Eq. (4.71), we see that $f(\alpha;s)$ may be represented as

$$f(\alpha;s) = \pi\,\mathrm{Tr}[\hat{f}(\hat{a},\hat{a}^\dagger)\hat{\delta}(\hat{a}-\alpha;-s)]. \quad (4.77)$$

Equation (4.77) may be viewed as the sought prescription for calculating the c-number function $f(\alpha;s)$ associated with the operator $\hat{f}(\hat{a},\hat{a}^\dagger)$ in s order from $\hat{f}(\hat{a},\hat{a}^\dagger)$ via the s-ordered delta operator. Substitution of the expression (4.77) into Eq. (4.50) yields the operator expansion in the phase space

$$\hat{f}(\hat{a},\hat{a}^\dagger) = \pi\int \mathrm{d}^2\alpha\;\mathrm{Tr}[\hat{f}(\hat{a},\hat{a}^\dagger)\hat{\delta}(\hat{a}-\alpha;-s)]\hat{\delta}(\hat{a}-\alpha;s). \quad (4.78)$$

Equivalently, we may expand $\hat{f}(\hat{a},\hat{a}^\dagger)$ in terms of the s-ordered displacement operator $\hat{D}(\alpha;s)$. Recalling Eq. (4.51), it is not difficult to see that Eq. (4.78) can be rewritten as

$$\hat{f}(\hat{a},\hat{a}^\dagger) = \frac{1}{\pi}\int \mathrm{d}^2\alpha\ \mathrm{Tr}[\hat{f}(\hat{a},\hat{a}^\dagger)\hat{D}(-\alpha;-s)]\hat{D}(\alpha;s). \tag{4.79}$$

4.3.2
The density operator in phase space

If we now identify in Eq. (4.78) [together with Eq. (4.77)] the operator $\hat{f}(\hat{a},\hat{a}^\dagger)$ with the density operator $\hat{\varrho}$, we obtain the following representation of the density operator:

$$\hat{\varrho} = \int \mathrm{d}^2\alpha\, \varrho(\alpha;s)\hat{\delta}(\hat{a}-\alpha;s), \tag{4.80}$$

where the c-number function associated with the density operator in s order reads

$$\varrho(\alpha;s) = \pi\,\mathrm{Tr}[\hat{\varrho}\hat{\delta}(\hat{a}-\alpha;-s)] = \pi\langle\hat{\delta}(\hat{a}-\alpha;-s)\rangle. \tag{4.81}$$

Comparing Eq. (4.81) with (4.53), we see that the phase-space function $P(\alpha;s)$ is (apart from the factor π^{-1}) identical to $\varrho(\alpha;-s)$,

$$P(\alpha;s) = \pi^{-1}\varrho(\alpha;-s). \tag{4.82}$$

Therefore, the density operator itself can be represented via Eq. (4.80) as $(s\to -s)$

$$\hat{\varrho} = \pi\int \mathrm{d}^2\alpha\, P(\alpha;s)\hat{\delta}(\hat{a}-\alpha;-s). \tag{4.83}$$

Note that from Eq. (4.83) the phase-space distribution $P(\alpha;s)$ may be seen explicitly to be normalized to unity,

$$\int \mathrm{d}^2\alpha\, P(\alpha;s) = 1, \tag{4.84}$$

because of $\mathrm{Tr}(\hat{\varrho})=1$ and Eq. (4.76).

In particular, from Eqs (4.52) and (4.82) we see that in calculating the expectation value of an operator $\hat{O}=\hat{f}(\hat{a},\hat{a}^\dagger)$ by "averaging" the c-number function $f(\alpha;1)\equiv f^{(\mathrm{N})}(\alpha,\alpha^*)$, which is associated with the operator $\hat{f}(\hat{a},\hat{a}^\dagger)$ put in normal order, the required phase-space function $P(\alpha;1)\equiv P(\alpha)$ is determined by the c-number function $\varrho(\alpha;-1)$ associated with the density operator $\hat{\varrho}$ put in anti-normal order (Glauber–Sudarshan representation), and vice versa. Only for symmetrical order $(s=0)$ are the c-number functions $f(\alpha;0)$ and $\varrho(\alpha;0)$

associated with the operators $\hat{f}(\hat{a},\hat{a}^\dagger)$ and $\hat{\varrho}$, respectively, both put into symmetrical order (Wigner representation).

Since expectation values of normally ordered operators are of particular importance in the context of quantities observable in optical photodetection experiments (Chapter 6), the Glauber–Sudarshan representation

$$\hat{\varrho} = \pi \int \mathrm{d}^2\alpha\, P(\alpha;1)\hat{\delta}(\hat{a}-\alpha;-1) \tag{4.85}$$

is often used in quantum optics. Substitution of the expression (4.66) into Eq. (4.85) yields $[P(\alpha) \equiv P(\alpha;1)]$

$$\hat{\varrho} = \int \mathrm{d}^2\alpha\, P(\alpha)|\alpha\rangle\langle\alpha|, \tag{4.86}$$

which is conceptually different from the straightforward representation of the density operator in terms of coherent states, as given in Eq. (4.17). The expectation value of an operator $\hat{O} = \hat{f}(\hat{a},\hat{a}^\dagger)$ may then be written with the help of Eq. (4.86) as

$$\begin{aligned}\langle \hat{f}(\hat{a},\hat{a}^\dagger)\rangle &= \int \mathrm{d}^2\alpha\, P(\alpha)\,\mathrm{Tr}[|\alpha\rangle\langle\alpha|\hat{f}(\hat{a},\hat{a}^\dagger)] \\ &= \int \mathrm{d}^2\alpha\, P(\alpha)\langle\alpha|\hat{f}(\hat{a},\hat{a}^\dagger)|\alpha\rangle,\end{aligned} \tag{4.87}$$

from which we also immediately recognize via Eq. (4.52) the result (4.23):

$$f(\alpha;1) = \langle\alpha|\hat{f}(\hat{a},\hat{a}^\dagger)|\alpha\rangle. \tag{4.88}$$

As already mentioned, $P(\alpha)$ can be highly singular. In particular, in the case of nonclassical states (Chapter 8), such as for example squeezed states, the calculation of $P(\alpha)$ may also lead to expressions that are not well behaved and are hard to interpret. However, using the phase-space representation defined by $\hat{\delta}(\hat{a}-\alpha;1)$,

$$\hat{\varrho} = \pi \int \mathrm{d}^2\alpha\, P(\alpha;-1)\hat{\delta}(\hat{a}-\alpha;1), \tag{4.89}$$

leads, according to Eq. (4.67), to the well-behaved Q function, $Q(\alpha) \equiv P(\alpha,-1) = \pi^{-1}\langle\alpha|\hat{\varrho}|\alpha\rangle$, which is suitable for the calculation of expectation values of anti-normally ordered operators.

We therefore observe a trade-off for the representation of the density operator in terms of phase-space functions that correspond to normal and anti-normal order. Either the phase-space function is well-behaved (Husimi Q function) and the associated delta operator may be problematic, or the phase-space function may be ill-behaved (Glauber–Sudarshan P function), whereas the delta operator is a simple projector on a coherent state. We note, however, that the ill-behaved and possibly singular expression do not formally represent serious obstacles for most derivations. Only concrete evaluations may turn out to be rather cumbersome.

The formal analogy between phase-space functions and classical statistics encourages the introduction of a formalism similar to that used in the usual probability theory. In particular, it is useful to introduce characteristic (generating) functions, by defining

$$\Phi(\alpha;s) = \langle \hat{D}(\alpha;s)\rangle. \tag{4.90}$$

According to Eqs (4.46) and (4.90), $\Phi(\alpha;s)$ and $\Phi(\alpha;s')$ are related to each other by

$$\Phi(\alpha;s) = \exp\left[\tfrac{1}{2}(s-s')|\alpha|^2\right]\Phi(\alpha;s'). \tag{4.91}$$

Equations (4.51), (4.53) and (4.90) reveal that, as usual, the characteristic function is the Fourier transform of the phase-space function, that is,

$$\Phi(\alpha;s) = \int \mathrm{d}^2\beta\, P(\beta;s)\exp(\alpha\beta^* - \alpha^*\beta), \tag{4.92}$$

and vice versa,

$$P(\alpha;s) = \frac{1}{\pi^2}\int \mathrm{d}^2\beta\, \Phi(\beta;s)\exp(\alpha\beta^* - \alpha^*\beta). \tag{4.93}$$

From the operator expansion (4.79) it can be seen that the characteristic function $\Phi(\alpha;s)$ also carries the full information about the quantum state. As in ordinary probability theory, $\Phi(\alpha;s)$ allows one to generate the various s-ordered moments $\langle \hat{a}^{\dagger k}\hat{a}^l\rangle_s$

$$\langle \hat{a}^{\dagger k}\hat{a}^l\rangle_s = \int \mathrm{d}^2\alpha\, \alpha^{*k}\alpha^l P(\alpha;s) = \frac{\partial^k}{\partial\beta^k}\frac{\partial^l}{\partial(-\beta^*)^l}\,\Phi(\beta;s)\Big|_{\beta=0}. \tag{4.94}$$

Here the s-ordered product $\{\hat{a}^{\dagger k}\hat{a}^l\}_s$ is defined by

$$\{\hat{a}^{\dagger k}\hat{a}^l\}_s = \frac{\partial^k}{\partial\alpha^k}\frac{\partial^l}{\partial(-\alpha^*)^l}\,\hat{D}(\alpha;s)\Big|_{\alpha=0}. \tag{4.95}$$

In Eq. (4.95) expressing $\hat{D}(\alpha;s)$ in terms of $\hat{D}(\alpha;s')$ according to Eq. (4.46) and differentiating, we may relate the s-ordered operator product to an s'-ordered operator product. After some algebra we derive

$$\{\hat{a}^{\dagger m}\hat{a}^n\}_s = \sum_{k=0}^{\min(m,n)} k!\binom{m}{k}\binom{n}{k}\left(\frac{s'-s}{2}\right)^k \{\hat{a}^{\dagger m-k}\hat{a}^{n-k}\}_{s'}. \tag{4.96}$$

The generalization of the concept of phase-space representation to multimode systems is straightforward. An extension of this concept to other than $\hat{a}$ and $\hat{a}^\dagger$ as basic operators is outlined in Section 5.2.2, in which the problem of formulating equations of motion (of Fokker–Planck type) for phase-space functions is studied. Moreover, we note that problems arising from singular behavior of the P function in the study of nonclassical states may be avoided by using generalized P representations [Drummond and Gardiner (1980); Gardiner (1983, 1991)],

$$\hat{\varrho} = \int_{\mathcal{D}} \mathrm{d}\mu(\alpha,\beta)\, \hat{\Lambda}(\alpha,\beta) P(\alpha,\beta), \tag{4.97}$$

where the operator $\hat{\Lambda}(\alpha,\beta)$ is given by

$$\hat{\Lambda}(\alpha,\beta) = \frac{|\alpha\rangle\langle\beta^*|}{\langle\beta^*|\alpha\rangle}, \tag{4.98}$$

and $\mathrm{d}\mu(\alpha,\beta)$ is the integration measure defining different classes of possible representations, with $\mathcal{D}$ being the domain of integration. In particular, it can be shown that the representation with measure

$$\mathrm{d}\mu(\alpha,\beta) = \mathrm{d}^2\alpha\, \mathrm{d}^2\beta \tag{4.99}$$

and integration over the whole complex plane always exists for a physical density operator and that $P(\alpha,\beta)$ can always be chosen positive, in which case it is called the positive P representation:

$$P(\alpha,\beta) = \frac{1}{4\pi^2}\, \exp\bigl(-\tfrac{1}{4}|\alpha-\beta^*|^2\bigr)\bigl\langle \tfrac{1}{2}(\alpha+\beta^*)|\hat{\varrho}|\tfrac{1}{2}(\alpha+\beta^*)\bigr\rangle. \tag{4.100}$$

4.3.3 Some elementary examples

To illustrate the theory, let us consider the Glauber–Sudarshan representation for some elementary quantum states, as introduced in Chapter 3. In the case of a coherent state $|\alpha_0\rangle$ we may immediately formulate the density operator as

$$\hat{\varrho} = |\alpha_0\rangle\langle\alpha_0| = \int \mathrm{d}^2\alpha\, \delta(\alpha-\alpha_0)\, |\alpha\rangle\langle\alpha|. \tag{4.101}$$

Comparing this equation with Eq. (4.86), we can obviously see that the Glauber–Sudarshan P function is

$$P(\alpha) = \delta(\alpha-\alpha_0). \tag{4.102}$$

From the point of view of classical statistics this function appears to have no fluctuations. In quantum theory such an interpretation is of course wrong.

From Sec. 3.2 we know that a system in a coherent state is indeed noisy, so that the appearance of a delta function in Eq. (4.102) should not mislead one to associate with it a deterministic behavior of the quantum system. In fact, this is only an effect of the chosen operator order, since other distributions with $s<1$ reveal a nonvanishing variance around the value α_0. We may furnish this, for example, by calculating the variance of the excitation number $\hat{n}=\hat{a}^\dagger\hat{a}$. To apply the Glauber–Sudarshan representation it is necessary to put the operator $(\Delta\hat{n})^2$ in normal order,

$$(\Delta\hat{n})^2 = (\hat{a}^\dagger\hat{a})^2 - \langle\hat{a}^\dagger\hat{a}\rangle^2 = \hat{a}^{\dagger 2}\hat{a}^2 + \hat{a}^\dagger\hat{a} - \langle\hat{a}^\dagger\hat{a}\rangle^2, \tag{4.103}$$

The evaluation of the expectation value of Eq. (4.103) is then performed as follows:

$$\langle(\Delta\hat{n})^2\rangle = \int \mathrm{d}^2\alpha\, P(\alpha)(|\alpha|^4+|\alpha|^2) - \left[\int \mathrm{d}^2\alpha\, P(\alpha)|\alpha|^2\right]^2, \tag{4.104}$$

which with $P(\alpha)=\delta(\alpha-\alpha_0)$ [Eq. (4.102)] just leads to the familiar result $\langle(\Delta\hat{n})^2\rangle=\langle\hat{n}\rangle=|\alpha_0|^2$. Clearly, the normally ordered variance $\langle :(\Delta\hat{n})^2:\rangle$, and generally any normally ordered moment of a mean-value deviation, vanishes in the case of the δ peaked distribution function.

Next, let us consider a thermal state. In the case of radiation a thermal state serves as an example of so-called chaotic light. As is well known, if a harmonic oscillator of frequency ω is in thermal equilibrium with a heat bath of temperature T, the density operator $\hat{\varrho}$ may be written in the form

$$\hat{\varrho} = \frac{\exp[-\hbar\omega\hat{n}/(k_\mathrm{B}T)]}{\mathrm{Tr}\{\exp[-\hbar\omega\hat{n}/(k_\mathrm{B}T)]\}} = \frac{1}{n_\mathrm{th}+1}\left(\frac{n_\mathrm{th}+1}{n_\mathrm{th}}\right)^{-\hat{n}}, \tag{4.105}$$

where the mean number of thermal photons, $n_\mathrm{th}=\langle\hat{n}\rangle$, is given by the familiar formula

$$n_\mathrm{th} = \left[\exp\left(\frac{\hbar\omega}{k_\mathrm{B}T}\right) - 1\right]^{-1}. \tag{4.106}$$

Note that for a thermal state the mean coherent amplitude vanishes: $\langle\hat{a}\rangle=0$. To calculate $P(\alpha)$, we recall that $P(\alpha)$ is determined by the c-number function $\varrho(\alpha;-1)$ associated with the density operator $\hat{\varrho}$ put in anti-normal order [Eq. (4.82) with $s=1$]:

$$P(\alpha) = \pi^{-1}\varrho(\alpha;-1). \tag{4.107}$$

To put $\hat{\varrho}$ in anti-normal order, we note that, after a straightforward but somewhat lengthy calculation [using, e. g., Eq. (4.96)], the anti-normally ordered form of an exponential operator $\exp(-z\hat{a}^\dagger\hat{a})$ may be written as

$$e^{-\hat{a}^\dagger\hat{a}z} = e^z \sum_{k=0}^{\infty} \frac{(1-e^z)^k}{k!}\,\hat{a}^k\hat{a}^{\dagger k}. \tag{4.108}$$

From Eqs (4.105) and (4.108) the anti-normally ordered form of $\hat{\varrho}$ is then found to be

$$\hat{\varrho} = \frac{1}{n_{\mathrm{th}}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\hat{a}^k \hat{a}^{\dagger k}}{n_{\mathrm{th}}^k}, \tag{4.109}$$

so that we obtain for the associated c-number function

$$\varrho(\alpha; -1) = \frac{1}{n_{\mathrm{th}}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{|\alpha|^2}{n_{\mathrm{th}}} \right)^k = \frac{1}{n_{\mathrm{th}}} \exp\left(-\frac{|\alpha|^2}{n_{\mathrm{th}}} \right). \tag{4.110}$$

Hence in this case the P function is a well-behaved Gaussian:

$$P(\alpha) = \frac{1}{\pi n_{\mathrm{th}}} \exp\left(-\frac{|\alpha|^2}{n_{\mathrm{th}}} \right). \tag{4.111}$$

In the case of radiation one may also think of the more general situation where the thermal light of mean photon number n_{th} is superimposed on coherent light of (complex) amplitude α_0, so that $\langle \hat{n} \rangle = n_{\mathrm{th}} + |\alpha_0|^2$ and $\langle \hat{a} \rangle = \alpha_0 \neq 0$. The superposition of the thermal state by a coherent one may be represented by a displacement of the distribution in phase space by the coherent amplitude α_0. The displaced P function is then obtained as

$$P(\alpha) = \frac{1}{\pi n_{\mathrm{th}}} \exp\left(-\frac{|\alpha - \alpha_0|^2}{n_{\mathrm{th}}} \right), \tag{4.112}$$

which obviously corresponds to the density operator

$$\begin{aligned} \hat{\varrho} &= \hat{D}(\alpha_0) \frac{1}{n_{\mathrm{th}} + 1} \left(\frac{n_{\mathrm{th}} + 1}{n_{\mathrm{th}}} \right)^{-\hat{a}^\dagger \hat{a}} \hat{D}^\dagger(\alpha_0) \\ &= \frac{1}{n_{\mathrm{th}} + 1} \left(\frac{n_{\mathrm{th}} + 1}{n_{\mathrm{th}}} \right)^{-(\hat{a}^\dagger - \alpha_0^*)(\hat{a} - \alpha_0)}. \end{aligned} \tag{4.113}$$

Note that, for coherent light, $n_{\mathrm{th}} = 0$, Eq. (4.112) reduces to the P function in Eq. (4.102), whereas for chaotic light, $\alpha_0 = 0$, it reduces to Eq. (4.111). The superposition of chaotic light with coherent light may be viewed as a simple model for characterizing the properties of single-mode laser light.

The above P functions exhibit all the properties of ordinary probability distribution functions. The only difference from classical statistics is that they apply to the calculation of normally ordered expectation values only. Since negative values of normally ordered variances, which are related to negative values of $P(\alpha)$, indicate nonclassical states (Chapter 8), quantum states such as coherent states or thermal states with a well-behaved P function may therefore be said to have a classical analog.

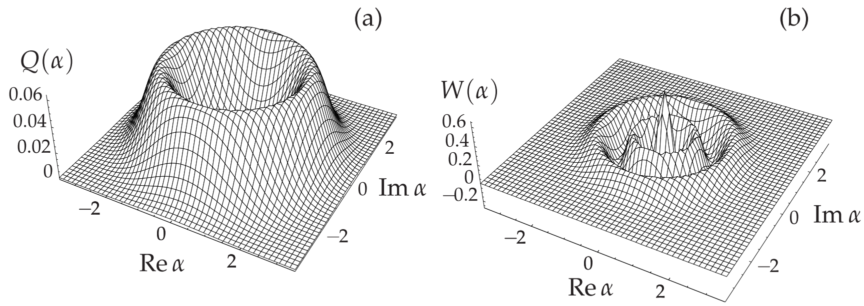

Fig. 4.1 Phase-space functions for the number state $|n=4\rangle$. Part (a) shows the Q function and part (b) the Wigner function, which reveals additional oscillatory and partially negative contributions.

As an example of a typical quantum state having no classical analog let us consider a number state

$$\hat{\varrho} = |n\rangle\langle n|. \tag{4.114}$$

To calculate the P function, we apply Eq. (4.53) directly and use Eq. (4.51) to obtain

$$P(\alpha) = \frac{1}{\pi^2}\int \mathrm{d}^2\beta\, \exp(\alpha\beta^* - \alpha^*\beta)\langle n|\hat{D}(\beta;1)|n\rangle, \tag{4.115}$$

where $\langle n|\hat{D}(\beta;1)|n\rangle$ can be calculated as

$$\langle n|\hat{D}(\beta;1)|n\rangle = \langle n|e^{\beta\hat{a}^\dagger}e^{-\beta^*\hat{a}}|n\rangle = \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k!}\,|\beta|^{2k}. \tag{4.116}$$

Inserting this result into Eq. (4.115), reproducing the factors $|\beta|^{2k}$ by derivatives with respect to α and α^* and performing the remaining integration, we obtain the P function as

$$P(\alpha) = \sum_{k=0}^{n}\binom{n}{k}\frac{1}{k!}\,\frac{\partial^k}{\partial\alpha^k}\,\frac{\partial^k}{\partial\alpha^{*k}}\,\delta(\alpha). \tag{4.117}$$

As expected, $P(\alpha)$ is highly singular and bears no resemblance to a proper probability distribution function. Nevertheless, it may be used to calculate normally ordered expectation values.

The Wigner function $W(\alpha)$ can be obtained from Eq. (4.115) by replacing therein $\hat{D}(\beta;1)$ with $\hat{D}(\beta) = \hat{D}(\beta;0)$ [cf. Eqs (4.51) and (4.53)]. Recalling Eq. (4.45), we may therefore write

$$W(\alpha) = \frac{1}{\pi^2}\int \mathrm{d}^2\beta\, \exp\big(\alpha\beta^* - \alpha^*\beta - \tfrac{1}{2}|\beta|^2\big)\langle n|\hat{D}(\beta;1)|n\rangle. \tag{4.118}$$

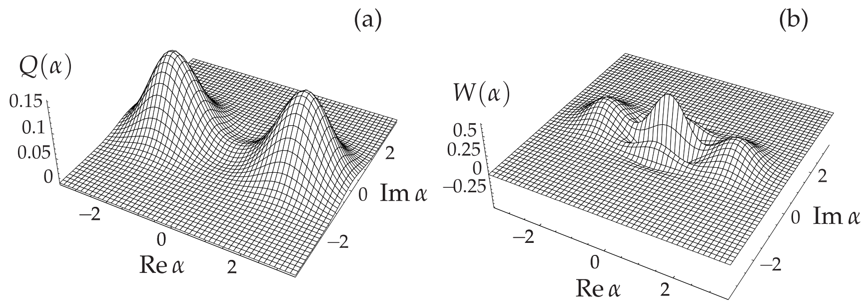

Fig. 4.2 Phase-space functions for a superposition of two coherent states $|\psi\rangle = N(|\alpha\rangle + |-\alpha\rangle)$ with $\alpha = 2$. Compared with the Q function (a) the Wigner function (b) shows negative values between the peaks of the two coherent states, which are signatures of their mutual quantum interference.

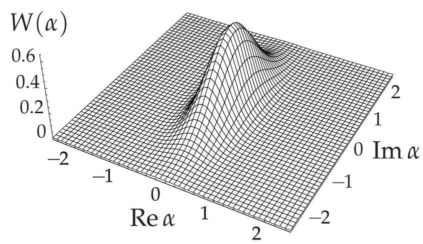

Fig. 4.3 Wigner function for a squeezed ground state, i. e., $|\beta, \xi\rangle$ where $\beta = 0$, and the squeezing parameter is $\xi = 0.5$.

Substituting the expansion as given by Eq. (4.116) for $\langle n|\hat{D}(\beta;1)|n\rangle$, we can calculate the β-integral to obtain

$$W(\alpha) = 2\pi^{-1}(-1)^n e^{-2|\alpha|^2} \mathrm{L}_n(4|\alpha|^2) \tag{4.119}$$

($\mathrm{L}_n(x)$ is the Laguerre polynomial). Since $\mathrm{L}_n(4|\alpha|^2)$ can take positive and negative values, the Wigner function – although well behaved – takes positive and negative values as well. As already mentioned, the Q function, $Q(\alpha) \equiv P(\alpha; -1)$, is well behaved and positive in any case. In the above example, with the system being in a number state, we easily find that

$$Q(\alpha) = \frac{1}{\pi}\,|\langle\alpha|n\rangle|^2 = \frac{1}{\pi}\,\frac{|\alpha|^{2n}}{n!}\,e^{-|\alpha|^2}. \tag{4.120}$$

Examples of the Q function for a number state and a superposition of two coherent states are shown in Figs 4.1(a) and 4.2(a), respectively. Their Wigner function counterparts are shown in Fig. 4.1(b) for the number state and in Fig. 4.2(b) for the coherent-state superposition. It is clearly observable that, in general, the Wigner function reveals sharper structures as compared with the Q function. Moreover, negative values occur in the Wigner function, being a signature of quantum interference effects. The Wigner function of a squeezed ground state is shown in Fig. 4.3.

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